Christ Church Grammar School

2017
Investigation 3

MATHEMATICS METHODS Year 12

Part One:
Take home investigation

Student name ___________________________________

Teacher name ___________________________________

Time allowed for this section

Working time for this section: 2 weeks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, and up to three calculators approved for use in the WACE examinations

Important note to candidates

This investigation has 2 parts: a take home investigation that the student needs to research and complete at home, and an in-class validation that will be completed in test conditions during class.

The take home investigation will not be assessed but has to be completed prior to the validation. Solution for the take home investigation will be provided.

The in-class validation will be assessed and it constitutes **7%** of the course mark.

This paper is the take home investigation.
It is handed out on Tuesday 27 June 2017.

The in-class validation will be on Thursday 20 July 2017.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

2. Answer all questions.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that **you do not use pencil**, except in diagrams.

THE NORMAL DISTRIBUTION

Aim of Investigation

The aim of this investigation is to examine events whose probabilities can be modelled by a Normal Distribution (bell shape), learn how to calculate the probabilities and quantiles in a Normal Distribution and use them to solve practical problems.

Learning Objectives

At the end of this investigation, you should be able to:

- identify contexts, such as naturally occurring variation, that are suitable for modelling by normal random variables
- identify features of the graph of the probability density function of the normal distribution with mean and standard deviation and the use of the standard normal distribution
- calculate probabilities and quantiles associated with a given normal distribution using technology, and use these to solve practical problems

Required Material

1. All the material contained in this booklet.
2. All the material found in:
   A.J. Sadler, Mathematics Methods Unit 4, Chapter 4 The Normal Distribution

Instructions to Candidates

1. To achieve the objectives of this investigation and to prepare for the validation, students will have to work through all the notes and questions specified as Required Material.
2. Additional resources can be found in:
   O.T. Lee, WACE Revision Series, Mathematics Methods Year 12, Pages 203 – 210.
   Creelman Exam Questions, Mathematics Methods Units 3 & 4, Pages 117 – 123.

Investigation

In an agricultural sciences study, five hundred 10-acre plots of land were prepared and tended in identical fashion for a wheat crop. At harvest, the size of the crop yield of each plot was recorded in bushels.

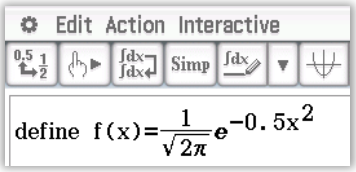
The crop yield of the plots was found to be normally distributed with a mean of 3.9 bushels and a standard deviation of 0.45 bushels.

Question 1

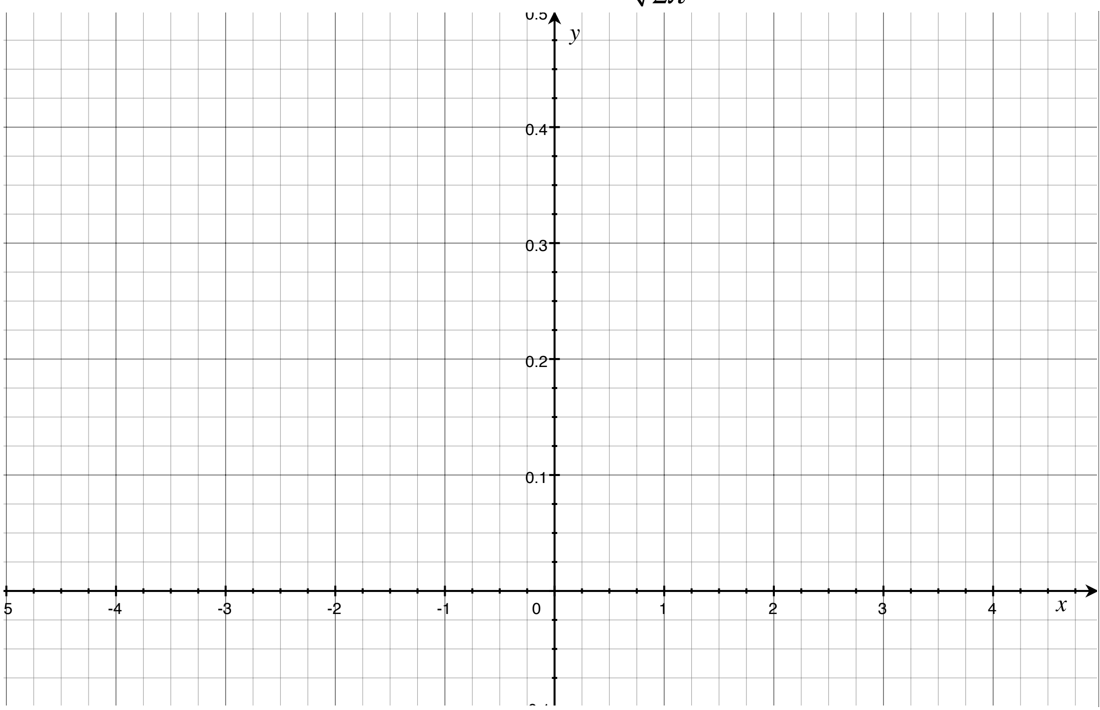
Consider the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}$

This is equation of the graph of the standard normal distribution with mean = 0 and standard deviation = 1

This function may be conveniently entered into a CAS calculator for convenient recall and calculation. For a Casio ClassPad the syntax is:



(a) Use your calculator to assist you to graph $f(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}$ on the axes below



(b) The graph of $f(x)$ is known as the **probability density function** for the standard normal distribution. Use your calculator to find the limit of $f(x)$ as $x \to \pm\infty$ to determine how the function behaves relative to the x-axis.

(c) With reference to your graph in part (a),
(i) Why is $x = 0$ significant?

(ii) Label the crop yields in bushels at the bottom of the graph in line with the corresponding standard deviations of the x-axis.

(d) What do you expect the area under the curve to be?

(e) Verify your answer in part (d) by evaluating $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}\,dx$ on your calculator.

Using the ClassPad, this can be done efficiently as $f(x)$ has already been defined. Use the syntax:

$$\int_{-\infty}^{\infty} f(x)\,dx$$

Similarly, determine $\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}\,dx$ and $\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}\,dx$

Explain this result with respect to part (c) & (d).

Question 2

(a) $f(x)=\dfrac{1}{\sqrt{2\pi}}e^{-0.5x^2}$ is a probability density function (PDF) with $\int_{-\infty}^{\infty} f(x)\,dx$ being the area under $f(x)$. Using this interpretation,

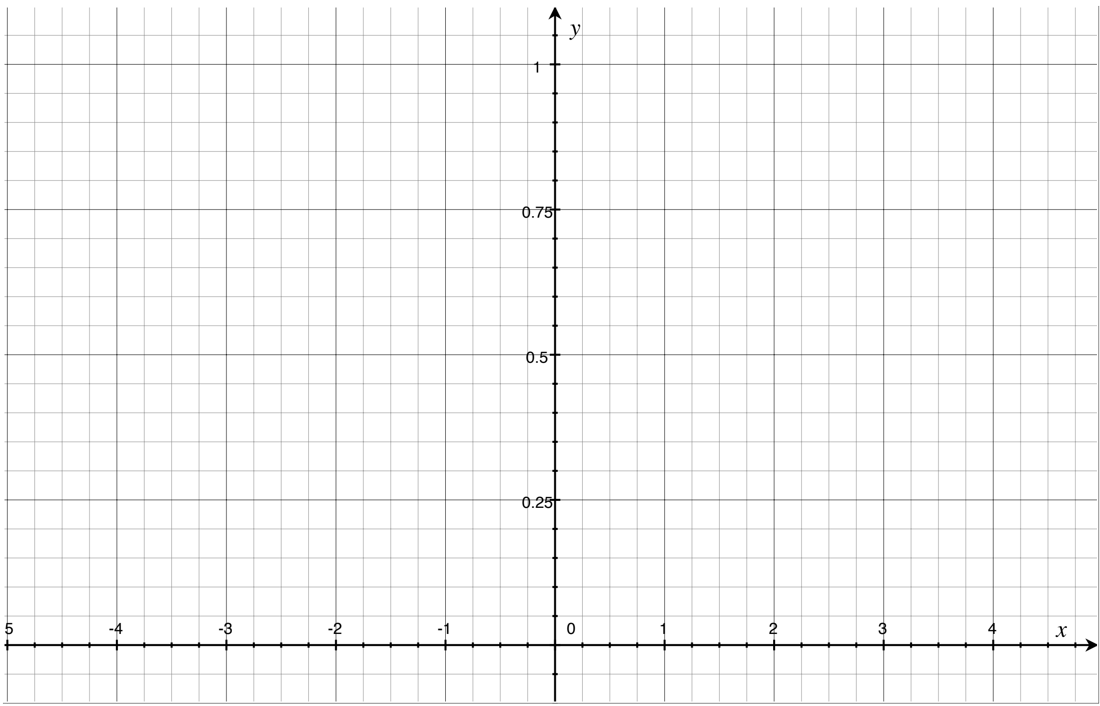
(i) explain the integral $\int_{-\infty}^{k} f(x)\,dx$ in terms of both area and probability.

(ii) explain what $P(x)=\int_{-\infty}^{x} f(t)\,dt$ represents.

(b) Define $P(x)=\int_{-\infty}^{x} f(t)dt$ on your calculator and

(i) graph $P(x)$ for $\{-3, -2.5, -2, \ldots 3\}$ on the axes below.
Note: this function is too demanding for the graphing features of your calculator.

(ii) show crop yields in bushels along the bottom of the graph



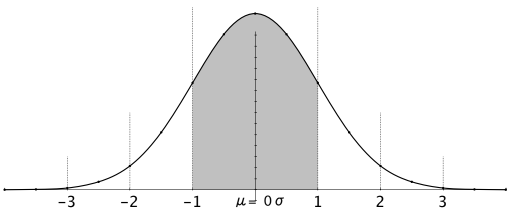
(c) This graph is a **cumulative distribution function** (CDF). Use limits to determine the behaviour of $P(x)$ as $x \to \pm\infty$ and explain this behaviour.

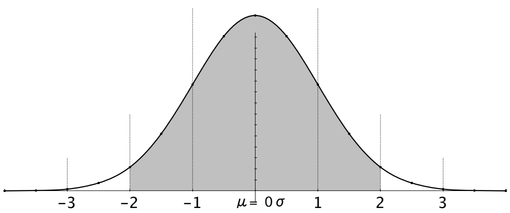
(d) Given that $a < b$, interpret $P(b)$, $P(a)$ and $P(b) - P(a)$.

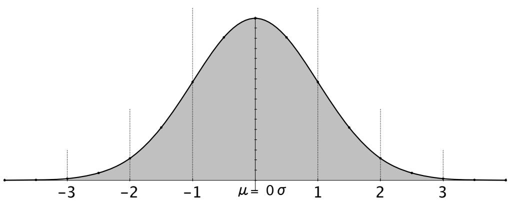
Question 3

Cumulative distributions effectively represent cumulative probabilities and these are commonly overlaid onto the probability density function.

(a) Use appropriate integrals to determine the proportions of the shaded regions for the PDF's shown below.







(b) Research and explain briefly the $68 - 95 - 99.7$ rule.

Question 4

When considering the crop yields we can think about the yields in one of two ways:

- The raw score measurements, i.e. a particular quantity of bushels harvested. These are known as the ***X-scores***.
- The number of standard deviations a particular quantity is above or below the mean. These are known as the ***z-scores***.

This equivalence will be reinforced each time you scale the crop yield (***X-scores***) against the standard deviations (***z-scores***) in Questions 4 & 5. Converting from the *X-scores* to the z-scores is known as **standardizing** the scores.

You have already performed these calculations informally. For example, in the case of our wheat crops we know

$$\mu = 3.9\, bushels$$
$$\sigma = 0.45\, bushels$$

So a yield of

$$\begin{aligned} 4.8\, bushels &= 3.9 + 2(0.45) \\ &= \mu + 2\sigma \\ 3.45\, bushels &= 3.9 - 1(0.45) \\ &= \mu - \sigma \end{aligned}$$

Thus, $X-score = 4.8 \Rightarrow z-score = 2$ i.e two S.D.'s **above** the mean (μ).

Whilst, $X-score = 3.45 \Rightarrow z-score = -1$ i.e. one S.D. **below** the mean (μ).

(a) Determine and/or research the formula to convert from an *X-score* to a *z-score*

(b) Convert the following *X-scores* to *z-scores* using $\mu = 3.9$ & $\sigma = 0.45$.

(i) $X = 4.125\, bushels$

(ii) $X = 3.225\, bushels$

(iii) $X = 2.550\, bushels$

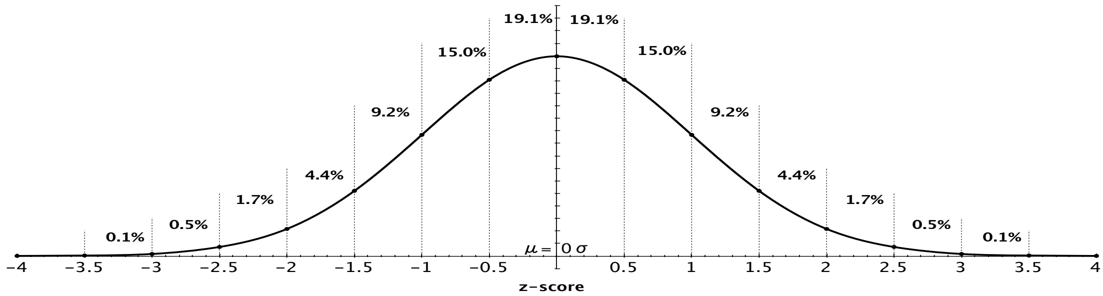
(iv) $X = 5.025\, bushels$

Question 5

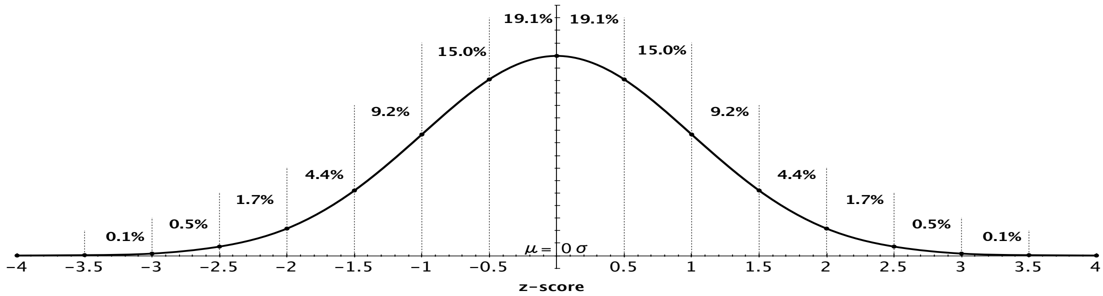
The $68 - 95 - 99.7$ rule is obtained by considering *z-scores* to the nearest whole standard deviation. In this question we will refine it down to the nearest ½ σ.

Using the *z-scores* calculated in Question 4, ***shade and state*** the proportion of the distribution that is:

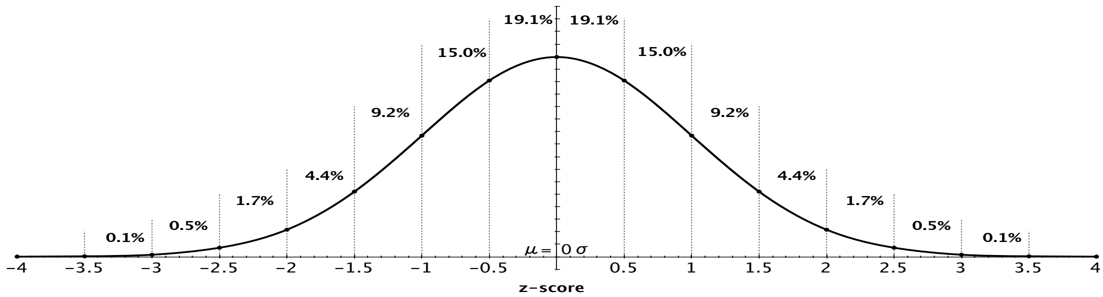
(a) "**Left-tailed**"
when $X < 4.125$



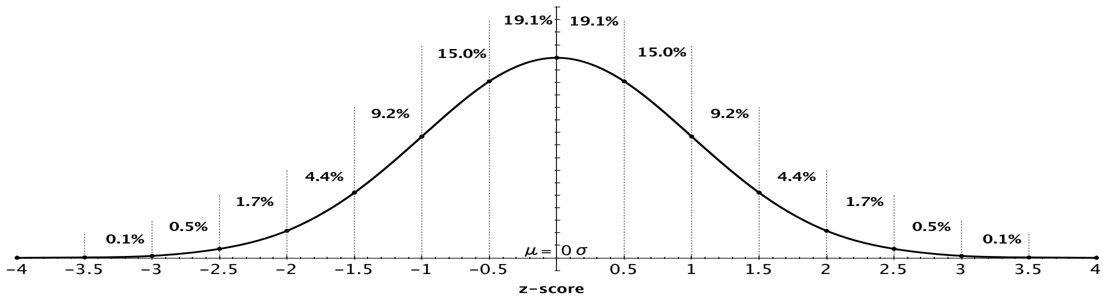
(b) "**Right-tailed**"
when $X > 3.225$



(c) "**Centred**"
when $3.675 < X < 4.125$



(d) "**Bounded**"
when $3.450 < X < 4.575$



(e) Two crop yields have their $z-scores$ calculated as $z=-1$ and $z=1.5$.

What is the probability of:

(i) a harvest yield of $z-score < -1$.

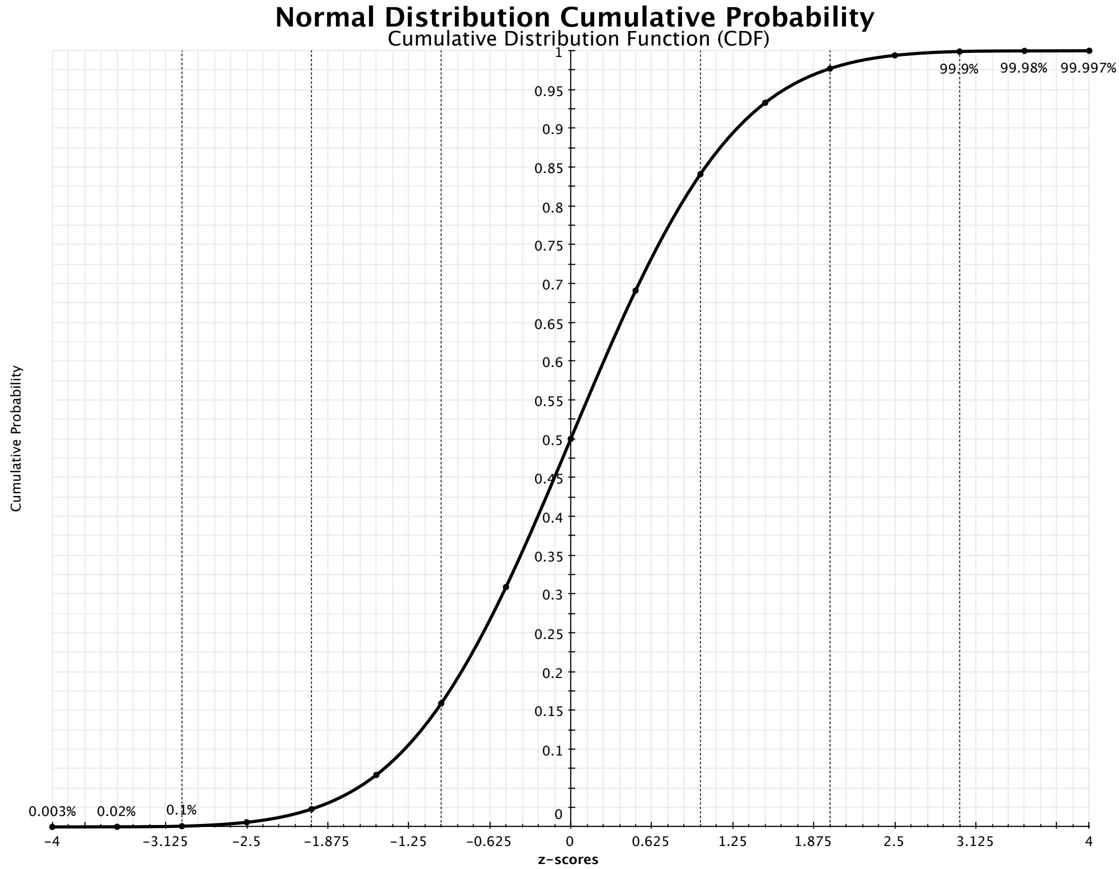
(ii) a harvest yield of $z-score > 1.5$.

(iii) a harvest yield with $-1 < z-score < 1.5$.

Determine the actual yields (i.e. *X-scores*) in bushels for each $z-score$.

Using The Cumulative Distribution Function

Determining the *Cumulative Distribution Function (CDF)*. beyond the nearest ½ S.D. requires a more accurate chart, or the use of task specific features on a CAS calculator.



Question 6
Graphical: z-score to probability (x-axis → y-axis)

(a) Label the *X-scores* in intervals of 1 standard deviation across the top of the above graph.

(b) For each of the following questions, draw construction lines on the CDF above to show your solution.

What would the probability be of a crop harvest yielding the following?

(i) $z-score < -1.25$

(ii) $z-score < 1.875$

(ii) $-1.25 < z-score < 1.875$

Question 7
CAS Calculator: z-score to probability (x-axis → y-axis)

$P(x)=\int_{-\infty}^{x} f(t)\,dt$ where $f(x)=\frac{1}{\sqrt{2\pi}}e^{-0.5x^2}$ is computationally intensive. So CAS calculators have specialized functions to handle these calculations more efficiently and automatically standardize scores, using the cumulative distribution function to calculate left & right tailed and centred cumulative probabilities.

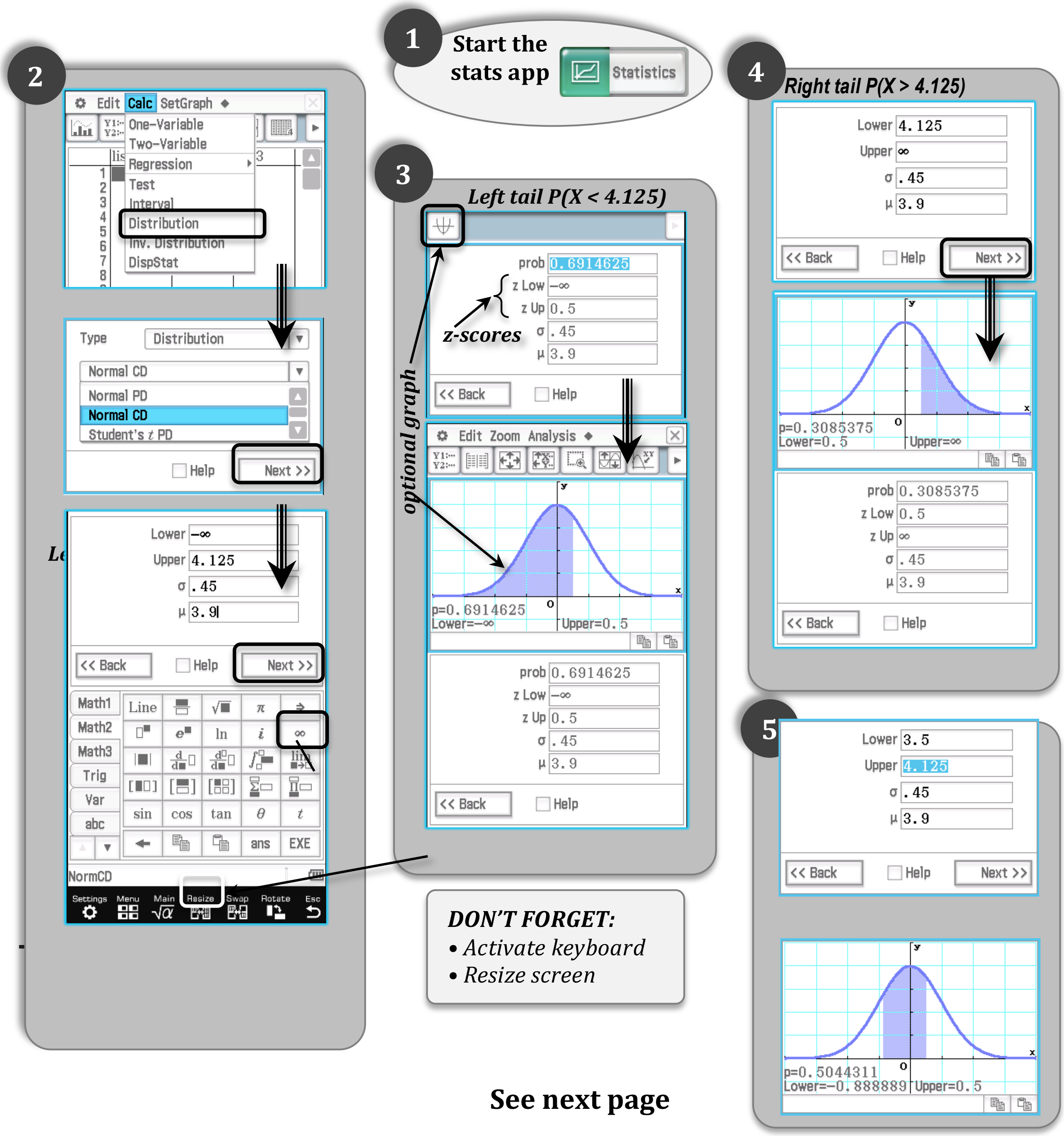
Notation: To indicate that a variable *score* X is part of a normal distribution we use the notation: $X\sim\left(\mu,\ \sigma^2\right)$.

Thus for our data set with $\mu=3.9$ and $\sigma=0.45$ we write $X\sim\left(3.9,\ 0.45^2\right)$

For example: To determine the proportion of scores below $X=4.125$ we write

$$P(X<4.125) \qquad \text{where} \qquad X\sim\left(3.9,\ 0.45^2\right)$$

Thus on a ClassPad calculator: $P(X<4.125)$ - a "left-tailed" cumulative probability - is found thus:



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Use your calculator to determine the following (to 4 d.p.)

(a) $P(X < 3.4)$

(b) $P(X > 2.9)$

(c) $P(2.4 < X < 3.1)$

Question 8
Graphical: Probability to z-score (y-axis → x-axis)
Refer back to the graph of the cumulative distribution function (CDF) on page 11 to answer the following questions. Draw construction lines on the CDF for each question to show your solution.

(a) A farmer determines that lowest 35% of harvests are not financially viable.
   (i) What $z-score$ does this correspond to?

   (ii) Hence, what is the minimum harvest (in bushels) to be profitable?

(b) The top 10% of harvests are considered to be too improbable for use in calculating a projected income range.
   (i) What is the maximum harvest (in bushels) that can be considered?

   (ii) Given that wheat fetches \$8.50 per bushel, what would be the expected return (as a range) on a random plot of land?
   (Exclude the lowest 35% and the top 10% of the harvests.)

CAS Calculator: Probability to z-score (y-axis → x-axis)

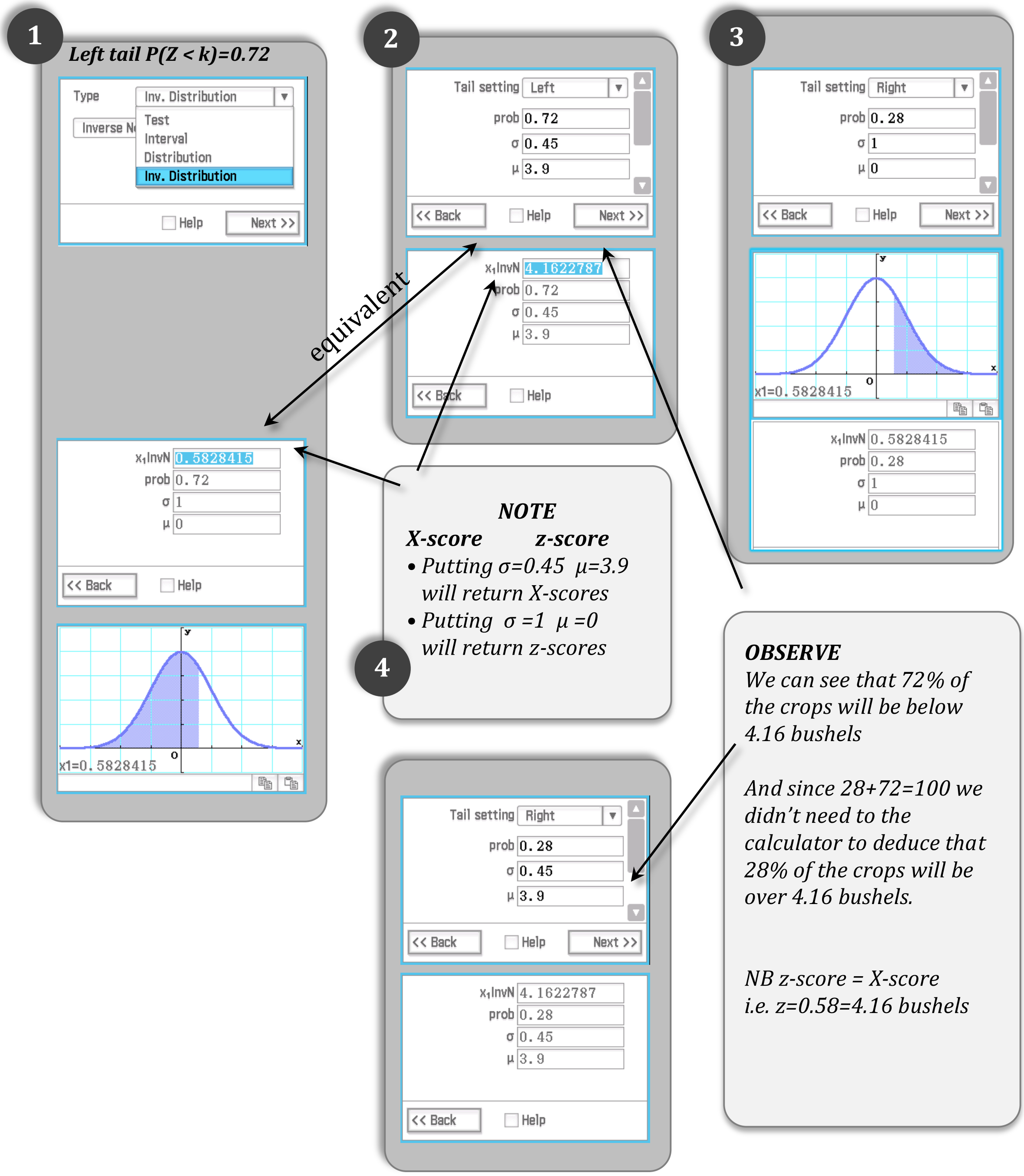
In Question 8 we needed to work out what *z-score* corresponded to a given cumulative probability. We did so by reading the *Normal Cumulative Distribution* off the *y-axis* and, using the *CDF* matched it up with a *z-score* on the *x-axis*.

Notation

Example: To find the *z-score* with a *Normal Cumulative Distribution* of 72%,

we write $P(z<k)=0.72$ where $X\sim(3.9, 0.45^2)$

k is the unknown *z-score* – note that this is a "left tailed" problem.



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Inverse Normal CD

Question 9
Use your calculator to confirm and obtain more accurate answers **(**to 4 d.p.) for Question 8.

(a) A farmer determines that the lowest 35% of harvests are not financially viable. What is the minimum harvest (in bushels) to be profitable?

(b) The top 10% of harvests are considered to be too improbable for use in calculating a projected income range. What is the maximum harvest (in bushels) that can be considered?

(c) Given that wheat fetches $8.50 per bushel,
(i) what would be the expected return (as a range) of a random plot of land?

(ii) From the 500 plots of land what would be the expected return in total?

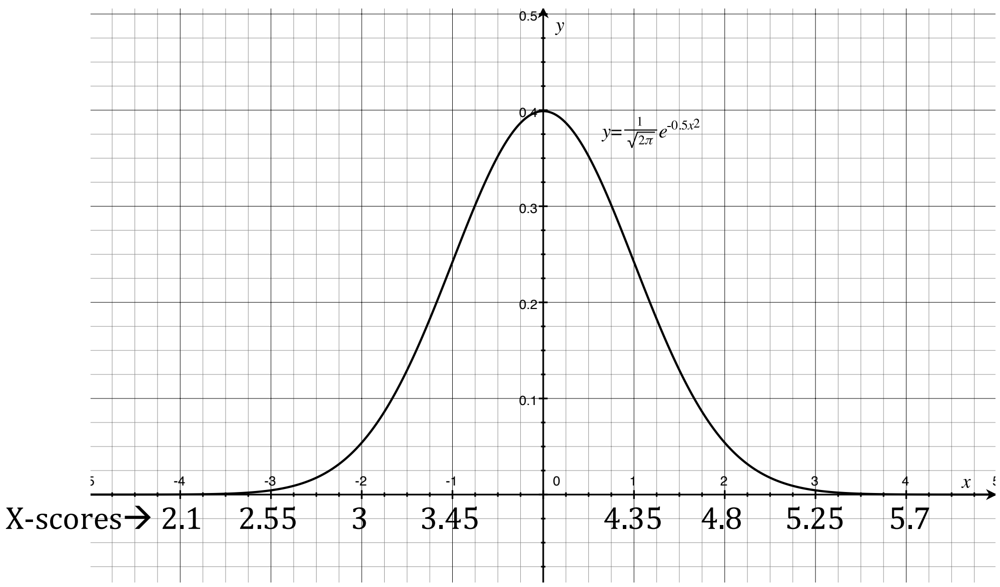
End of questions

Solutions

Question 1

Solution

(a)



(b) The curve is asymptotic to the x-axis, since:

$$\lim_{x\to\pm\infty} f(x)=0$$

$e^{-0.5x^2}>0$ so as $x\to+\infty \Rightarrow f(x)\to 0^+$ i.e. from above

$e^{-0.5x^2}>0$ so as $x\to-\infty \Rightarrow f(x)\to 0^+$ i.e. from above also

(c) (i) $x=0$ corresponds to the position of the mean. It is the centre of the distribution.

(ii) shown on the graph as X-scores

(d) The total area amounts to the total population as a proportion, i.e. 100% = 1.00.

(e) $x=0$ is the centre of the curve, in one case the upper boundary of the lower half, and in the other case the lower boundary of the upper half.

Total area is 1.0, so the area of each half will be 0.5.

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-0.5x^2}dx=1 \qquad \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}e^{-0.5x^2}dx=0.5 \qquad \int_{0}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-0.5x^2}dx=0.5$$

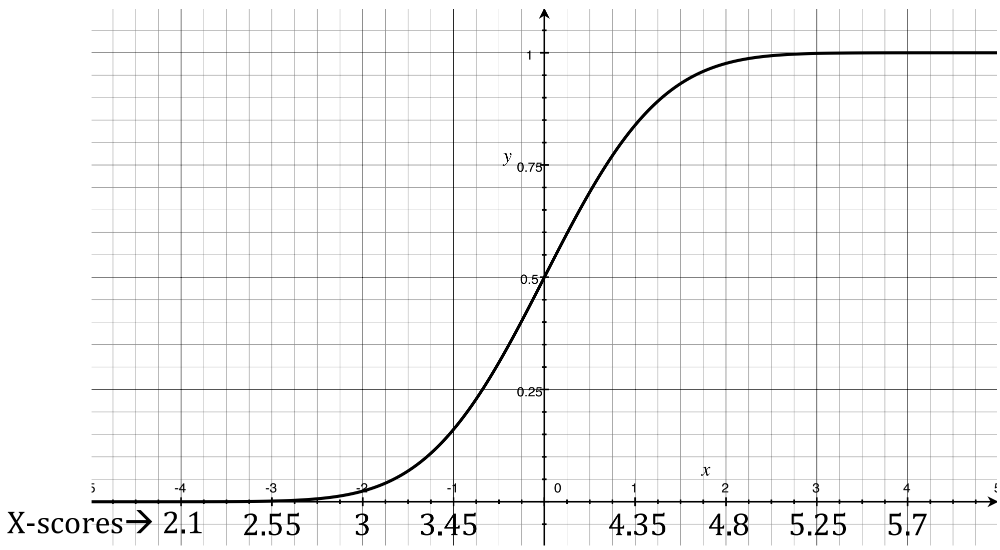
Question 2

Solution

(a) (i) This integral will calculate the area under the PDF from its lowest extreme up to the value k. Since the total area under the PDF = 1, this represents a proportion of the population and, hence, a percentage and a probability.

(ii) P(x) formulates a probability function that gives the probability of a score being in the area below the upper boundary of x.

(b) (i)



(ii)

(c)

As $x \to -\infty, \quad P(x) \to 0$ As $x \to -\infty$ the area bounded approaches zero

As $x \to +\infty, \quad P(x) \to 1$ As $x \to +\infty$ the area bounded apporaches the total area under the PDF curve = 1

(d) P(b) is the probability of a score being below 'b',
P(a), the probability of a score being below 'a' and
P(b) – P(a) the probability of a score falling within the area bounded at the lower end by 'a' and at the upper end by 'b'.

Question 3

Solution

(a) (i)

$$P(1) - P(-1) = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}\, dx = 0.6827$$

$$\approx 68.27\%$$

(ii)

$$P(2) - P(-2) = \int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}\, dx = 0.9545$$

$$\approx 95.45\%$$

(iii)

$$P(3) - P(-3) = \int_{-3}^{3} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}\, dx = 0.9973$$

$$\approx 99.73\%$$

(b) For a normally distributed population, 68% of the population will fall within ±1S.D. of the mean, 95% will fall within ±2 S.D's and 99.7% will fall within ±3 S.D.'s

Question 4

Solution

(a)

$$z - score = \frac{X - \mu}{\sigma}$$

(b) (i)

$$z = \frac{4.125 - 3.9}{0.45} \quad \Rightarrow \quad z = 0.5$$

(ii)

$$z = \frac{3.225 - 3.9}{0.45} \quad \Rightarrow \quad z = -1.5$$

(iii)

$$z = \frac{2.55 - 3.9}{0.45} \quad \Rightarrow \quad z = -3$$

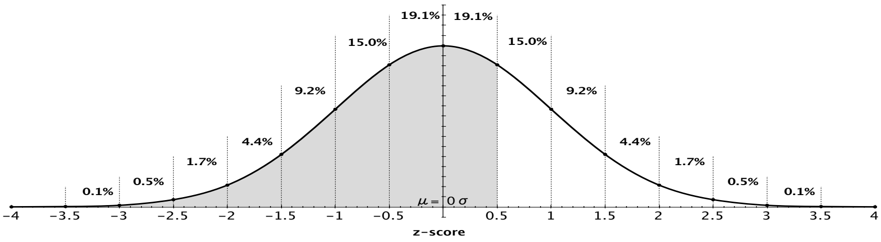
(iv)

$$z = \frac{5.025 - 3.9}{0.45} \quad \Rightarrow \quad z = 2.5$$

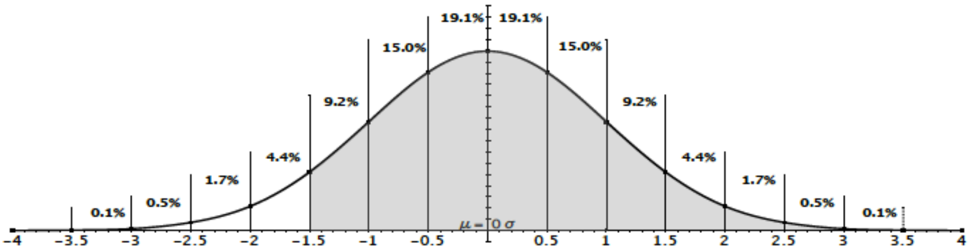
Question 5

Solution

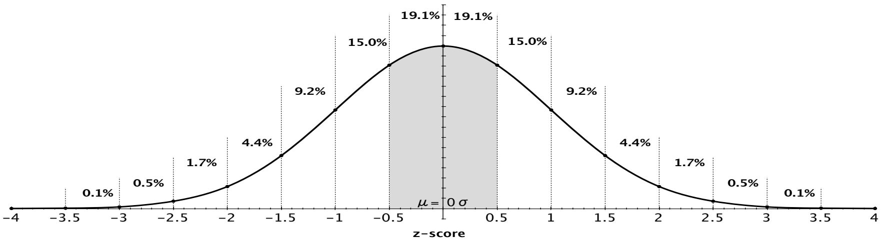
(a) i) *X < 4.125* *69.1%*



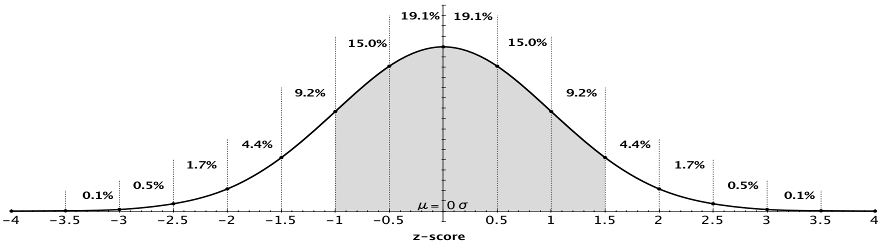
(b) *X > 3.225* *93.3%*



(c) *3.675 < X < 4.125* *38.2%*



(d) *3.450 < X < 4.575* *77.4%*



(e) (i) 15.9%

(ii) 6.7%

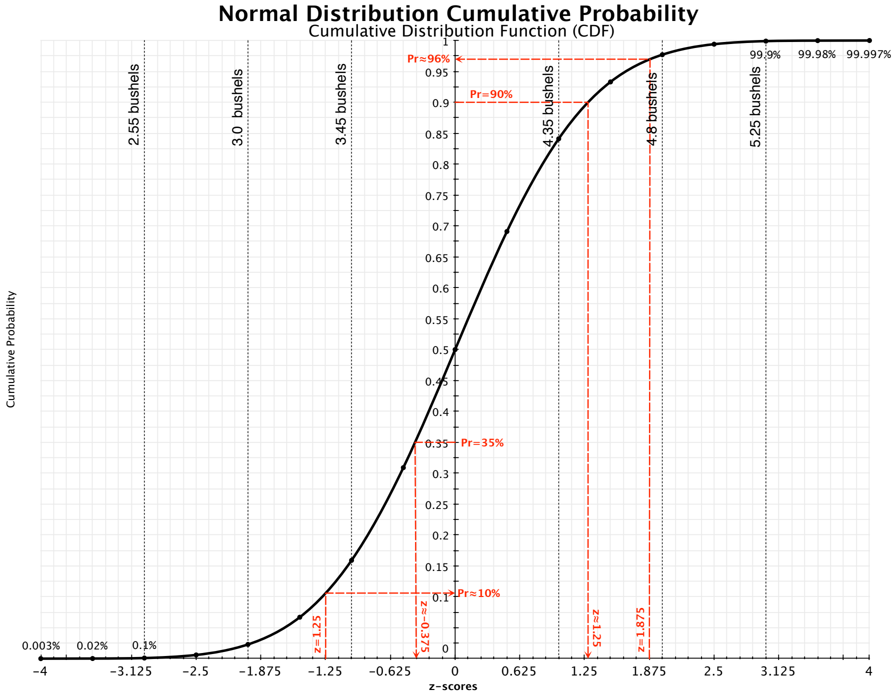
(iii) 77.4%

z=-1 → X=3.45 bushels
z=1.5 → X=4.575 bushels

Question 6

Solution

(a)



(b) (i) 10%

(ii) 96%

(iii) 96 – 10 = 86%

Question 7

Solution

(a) 0.1333

(b) 0.9869

(c) 0.0373

Question 8

Solution

(a) (i) $z \approx 0.375$

(ii)

$X - score = -0.375(-0.45) + 3.9$

$= 3.73125$

$\therefore$ Approx 3.73 bushels

(b) (i)

$z = 1.25 \quad \Rightarrow \quad X - score = 1.25(0.45 + 3.9)$

$= 4.4625$

$\therefore$ Approx 4.46 bushels

(ii)

$Min: \quad 3.73 \times 8.5 = 31.72$

$Max: \quad 4.46 \times 8.5 = 37.93$

$\therefore$ Expected return is from $31.70 to $37.90

Question 9

Solution

(a)

$P(x < k) = 0.35 \quad \Rightarrow \quad k = 3.7266$

$\therefore$ Minimum expected harvest is 3.73 bushels

(b)

$P(x > k) = 0.9 \quad \Rightarrow \quad k = 4.4767$

$\therefore$ Maximum expected harvest is 4.4767 bushels

(c) (i)

$Min: \quad 3.7266 \times 8.5 = 31.68$

$Max: \quad 4.4767 \times 8.5 = 38.05$

$\therefore$ Expected return is from $31.70 to $38.10

(ii)

$Min: \quad 500 \times 31.68 = 19025.97$

$Max: \quad 500 \times 38.05 = 15838.09$

$\therefore$ Expected return is from $15 800 to $19 000